Outline: Curvature

1. **The Angle Function**

   An **angle function** for a curve $\vec{x}(t)$ is a continuous function $\theta(t)$ such that

   $$\vec{T}(t) \;=\; \big(\cos\theta(t), \sin\theta(t)\big)$$

   where $\vec{T}(t)$ is the unit tangent vector. The angle function for a curve is uniquely determined up to adding or subtracting integer multiples of 2π.

2. **Curvature**

   For a unit-speed curve $\vec{x}(s)$, the **curvature** $\kappa_g(s)$ is defined by

   $$\kappa_g(s) \;=\; \frac{d\theta}{ds} \;=\; \theta'(s)$$

   where $\theta(s)$ is an angle function for the curve. More generally, if $\vec{x}(t)$ is any curve (not necessarily unit speed), the curvature is given by the formula

   $$\kappa_g(t) \;=\; \frac{d\theta}{ds} \;=\; \frac{d\theta/dt}{ds/dt} \;=\; \frac{\theta'(t)}{s'(t)}$$

   where $\theta(t)$ is an angle function, and $s'(t)$ denotes the speed.

3. **Derivative of the Tangent Vector**

   If $\vec{x}(t)$ is a regular curve and $\vec{T}(t) = \big(\cos\theta(t), \sin\theta(t)\big)$ is its unit tangent vector, then

   $$\vec{T}'(t) \;=\; \theta'(t)\,\vec{U}(t),$$

   where $\vec{U}(t)$ is the unit normal vector. It follows that

   $$\vec{T}'(t) \;=\; \kappa_g(t)\,s'(t)\,\vec{U}(t).$$

4. **Rotating Unit Vectors**

   More generally, if $\vec{V}(t) = \big(\cos\theta(t), \sin\theta(t)\big)$ is any rotating unit vector, then

   $$\vec{V}'(t) \;=\; \theta'(t)\,\vec{U}(t)$$

   where $\vec{U}(t)$ is the unit vector whose direction is $90°$ counterclockwise from $\vec{V}(t)$.

5. **Osculating Circle**

   The **osculating circle** to a curve $\vec{x}(t)$ at time t is the circle with center $\vec{x}(t) + \dfrac{1}{\kappa_g(t)}\vec{U}(t)$ and radius $\big|1/\kappa_g(t)\big|$.

6. Acceleration

If $\vec{x}(t)$ is any parametric curve, then

$$\vec{x}\,''(t) \;=\; \kappa_g(t)\, s'(t)^2\, \vec{U}(t) \;+\; s''(t)\, \vec{T}(t).$$

Since $\vec{U}(t)$ and $\vec{V}(t)$ are perpendicular unit vectors, we can also write this formula as

$$\vec{x}\,''(t)\cdot\vec{U}(t) \;=\; \kappa_g(t)\, s'(t)^2 \qquad \text{and} \qquad \vec{x}\,''(t)\cdot\vec{T}(t) \;=\; s''(t).$$

The first of these equations can be used to find the curvature:

$$\kappa_g(t) \;=\; \frac{\vec{x}\,''(t)\cdot\vec{U}(t)}{s'(t)^2} \;=\; \frac{-x''(t)\,y'(t) + y''(t)\,x'(t)}{s'(t)^3}.$$

From a computational perspective, this is probably the most useful formula for curvature.

7. Derivatives of Vectors

More generally, if

$$\vec{V}(t) \;=\; r(t)\,\big(\cos\theta(t), \sin\theta(t)\big)$$

is any vector, then

$$\vec{V}\,'(t) \;=\; r(t)\,\theta'(t)\,\vec{U}(t) \;+\; r'(t)\,\vec{T}(t)$$

where $\vec{T}(t)$ is the unit vector in the direction of $\vec{V}(t)$, and $\vec{U}t$ is the unit vector $90°$ counterclockwise from $\vec{T}(t)$. The acceleration formula above is the case where $\vec{V}(t) = \vec{x}\,'(t)$ (and hence $r(t) = s'(t)$).

Since $\vec{T}(t)$ and $\vec{U}(t)$ are perpendicular vectors, this formula can also be written

$$\vec{V}\,'(t)\cdot\vec{U}(t) \;=\; r(t)\,\theta'(t) \qquad \text{and} \qquad \vec{V}\,'(t)\cdot\vec{T}(t) \;=\; r'(t).$$

8. The Integral of Curvature

If $\mathcal{C}$ is the curve $\vec{x}(t)$ for $a \le t \le b$, then

$$\int_{\mathcal{C}} \kappa_g(t)\, ds \;=\; \theta(b) - \theta(a).$$

If $\mathcal{C}$ is a regular closed curve, it follows that

$$\int_{\mathcal{C}} \kappa_g(t)\, ds \;=\; 2\pi n,$$

where n is an integer called the **rotation index** of $\mathcal{C}$.

9. Finding the Curve

Suppose we are given the curvature function $\kappa_g(s)$ for a unit-speed curve. Then the angle function $\theta(s)$ and parametrization $\vec{x}(s)$ can be recovered using the formulas

$$\theta(s) \;=\; \int \kappa_g(s)\, ds \qquad \text{and} \qquad \vec{x}(s) \;=\; \int \big(\cos\theta(s), \sin\theta(s)\big)\, ds.$$